

Proof that the real part of all non-trivial zeros of Riemann zeta function is 1/2

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This article proves the Riemann hypothesis, which states that all non-trivial zeros of the zeta function have a real part equal to $1/2$. We inspect in detail the integral form of the (symmetrized) completed zeta function, which is a product between the zeta and gamma functions. It is known that two integral lines, expressing the completed zeta function, rotated from the real axis in the opposite directions, can be shifted without affecting the completed zeta function owing to the residue theorem. The completed zeta function is regular in the region of the complex plane under consideration. For convenience in the subsequent singularity analysis of the above integral, we first transform (convert) this integral to that along the line parallel to the real axis. We then investigate the singularities of the composite elements (caused by polynomial integrals in opposite directions), which appear only in the case for which the distance between the contours and the origin of the coordinates approaches zero. The real part of the zeros of the zeta function is determined to be $1/2$ along a symmetry line from the singularity removal condition. (In the other points, the singularities are adequately cancelled as a whole to lead to a finite value.)

1 Introduction

By connecting complex analysis with number theory, Riemann observed [1] that (denoting a set of real numbers by \mathbb{R} and letting $x \in \mathbb{R}$) the function $\pi(x)$, which denotes the number of prime numbers below a given number x , contains the summation over non-trivial zeros (points at which the function vanishes) of the zeta function. Riemann expected (denoting a set of complex numbers by \mathbb{C} and letting $z \in \mathbb{C}$) the real part of the non-trivial zeros of the zeta function $\zeta(z)$ to be $1/2$, which is known as the Riemann hypothesis. Furthermore, von Koch showed [2] that $\pi(x)$ is well approximated by the offset logarithmic integral function $\text{Li}(x)$ as

$$\pi(x) = \text{Li}(x) + O(x^{\frac{1}{2}} \log x), \quad (1)$$

which is equivalent to the Riemann hypothesis. We denote a set of natural numbers by \mathbb{N} and let $n \in \mathbb{N}$ and $z \in \mathbb{C}$, then the zeta function $\zeta(z)$ is defined as a function, which is analytically continued in the complex plane from the expression defined below [3-5]

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad (2)$$

for z that satisfies $\text{Re}(z) > 1$ (we denote the real and imaginary parts of z as $\text{Re}(z)$ and $\text{Im}(z)$, respectively.) The zeta function is also obtained with the help of the gamma function $\Gamma(z)$, and, letting $t' \in \mathbb{R}$, then the gamma function is defined as a function that is also analytically continued into all points in the complex plane from [3,6-9]

$$\Gamma(z) := \int_0^{\infty} dt' (t')^{z-1} \exp(-t'), \quad (3)$$

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for $\text{Re}(z) > 0$.

Concerning the zeros of the zeta function, which states $\zeta(z) = 0$, there exist trivial zeros, such as negative integers $-2, -4, \dots$ [3]. In contrast, Hardy showed that numerous non-trivial zeros of the zeta function exist along the line with the real part equal to $1/2$ [10]; however, not all the real parts of the non-trivial zeros are known. The computational approach [11] strongly suggests that the real part of zeros of the zeta function is $1/2$.

On the other hand, letting $z, w \in \mathbb{C}$, for the completed zeta function defined by

$$\hat{\zeta}(z) := \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z), \quad (4)$$

integral form of the (completed) zeta function is expressed as [12]

$$\begin{aligned} & \pi^{-\frac{1-z}{2}} \Gamma\left(\frac{1-z}{2}\right) \zeta(1-z) \\ &= \pi^{-\frac{1-z}{2}} \Gamma\left(\frac{1-z}{2}\right) \int_{0 \searrow 1} dw \frac{w^{z-1} \exp(-\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)} \\ &+ \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \int_{0 \swarrow 1} dw \frac{w^{-z} \exp(\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)}. \end{aligned} \quad (5)$$

The above integral is performed along the integral lines $0 \searrow 1$ and $0 \swarrow 1$ with the slopes -1 and $+1$, respectively, which pass through an arbitrary point in the region between 0 and 1 of the real axis. Since the residue theorem exhibits the above equation, the integral form is independent of the shift of this intersection point between 0 and 1. Furthermore, in the original form [12] of the above equation, the function $\Gamma(z/2)$ in the second term on the right is proportional to the regular function for $\text{Re}(z) < 1$. The function $\Gamma((1-z)/2)$ on the left is regular in the region $\text{Re}(z) < 1$, while the right side is also regular because of the existence of the derivative [13, 14], and the function $\zeta(1-z)$ is analytically continued uniquely [13] into the region $0 < \text{Re}(z) < 1$ (the real part of zeros of $\zeta(z)$ exists only in this region). This paper takes into account the form mentioned above.

Since the gamma function is regular, the non-trivial zeros of the completed zeta function $\hat{\zeta}(z)$, which is the product between the gamma function $\Gamma(z)$ and zeta function $\zeta(z)$, coincide with those of the zeta function $\zeta(z)$ in the region being considered with the real part between 0 and 1. As is described in this paper, each of the two line integrals expressing the completed zeta function has a singularity when the integral lines approach the axis origin. However, the completed zeta function $\hat{\zeta}(z)$ does not depend on a specific point of the intersection point (shifted between 0 and 1 along the real axis) between the above integral line and the real axis due to the residue theorem, and $\hat{\zeta}(z)$ is regular in the considering region. Then, these singularities must exactly cancel each other for $\hat{\zeta}(z) = 0$, which is expected to lead to the determination of the real part of the zeros of the zeta function $\zeta(z)$.

Considering the status mentioned above, this paper is aimed at proving the Riemann hypothesis. We first transform (convert) the integral (in the integral form of the completed zeta function) along the integral line rotated from the real axis to the integral along the line parallel to the coordinate (real/imaginary) axis, for convenience in the subsequent analysis of the singularity of the integral in a complex plane. By this conversion of the integral, the singularity analysis can be concentrated on the component of the integral along the line parallel to the real axis (or the equivalent integral component), dropping the component of the integral in the direction of the imaginary axis.

This research then addresses the singularities that appear in the two integral lines of the integral form of the completed zeta function. The singularities of the integrands for the composite elements near the origin of the real axis are caused by polynomials, only in the case when the contour-origin distance approaches zero. These singularities adequately cancel each other yielding a finite value independent of the integral contour as a whole. In contrast, from the equation $\hat{\zeta}(z) = 0$ for completed zeta function $\hat{\zeta}(z)$, the real part of the zeros of $\hat{\zeta}(z)$ is determined by requiring these singularities to be an identical order power of the integral variable in the integrands leading to the exact singularity cancellation (given by Theorem 2). This requirement results in a value of $1/2$ for the real part of zeros of the completed zeta function $\hat{\zeta}(z)$ (and the original zeta function $\zeta(z)$) due to the symmetry with respect to the $1/2$ real part, which is the originality of the present study and proves the Riemann hypothesis. The Riemann hypothesis is one of the most

important unproved problems in mathematics, and has its equivalent and advanced (extended) conjectures in other related fields. The positive proof of the Riemann hypothesis advances mathematics in other related fields [15, 16].

The contents of this paper are as follows. Section 2 provides the transformation of the integral in the integral form of the completed zeta function to the line parallel to the real axis for convenience in the subsequent singularity analysis of the integral. Section 3 presents the proof that the real part of all non-trivial zeros of the zeta function is equal to $1/2$, as was conjectured by Riemann, followed by the conclusion.

2 Transformation of the integral to that along the line parallel to a coordinate axis for the singularity analysis in a complex plane

This section provides the transformation (conversion) of the integral (in the integral form of the completed zeta function) along the line rotated from the real axis to the integral along the line parallel to the real axis (or the equivalent integral component) for convenience in the subsequent analysis (in Section 3) of the singularity of the integral in a complex plane. By this transformation, the analysis of the integral in a complex plane becomes independent from the integral component in the direction of the imaginary axis. Notations used in this paper are as follows. Let $z \in \mathbb{C}$ and $x, y \in \mathbb{R}$ and let i be the imaginary unit, then

$$(x, y) := z = x + iy. \quad (6)$$

We denote the real and imaginary components as

$$z_{\mathbb{R}} := \operatorname{Re}(z) = x, \quad (7)$$

$$z_{\mathbb{I}} := \operatorname{Im}(z) = y. \quad (8)$$

We first define the integrands, the main points in the complex plane, and integrals with contours for use in the subsequent lemma in this section.

Definition 1. Let $w, z \in \mathbb{C}$, then the integrands $I_{\mathbb{N}}$ and $I_{\mathbb{P}}$ in w -plane are defined by

$$I_{\mathbb{N}} := \pi^{-\frac{1-z}{2}} \Gamma\left(\frac{1-z}{2}\right) \frac{w^{z-1} \exp(-\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)}, \quad (9)$$

$$I_{\mathbb{P}} := \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \frac{w^{-z} \exp(\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)}. \quad (10)$$

Definition 2. Let $a_0, a_{0\mathbb{I}}, a_{0\mathbb{R}} \in \mathbb{R}$ be positive finite numbers between 0 and 1, with $a_0 = a_{0\mathbb{I}}$ for $I_{\mathbb{N}}$ in Eq. (9) and $a_0 = a_{0\mathbb{R}}$ for $I_{\mathbb{P}}$ in Eq. (10), respectively. Let $b_0 \in \mathbb{R}$ with $b_0 > 0$ be a small number. Then we define the points $w_{\mathbb{I}\mathbb{U}}, w_{\mathbb{I}\mathbb{L}} \in \mathbb{C}$ in the complex plane, indicated in Fig. 1, as

$$w_{\mathbb{I}\mathbb{U}} := (a_0, b_0), \quad (11)$$

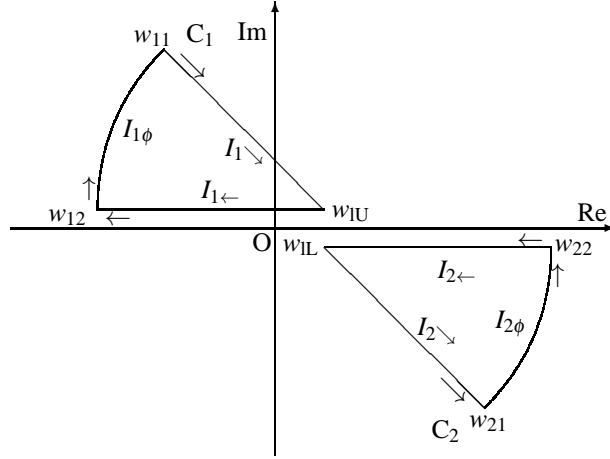
$$w_{\mathbb{I}\mathbb{L}} := (a_0, -b_0). \quad (12)$$

Let $\bar{\epsilon}_3 \in \mathbb{C}$ be defined by

$$\bar{\epsilon}_3 := \exp\left(\frac{3}{4}\pi i\right), \quad (13)$$

and let $r_2 \in \mathbb{R}$ with $r_2 > 0$ be the radius of arcs centered at $w_{\mathbb{I}\mathbb{U}}$ and $w_{\mathbb{I}\mathbb{L}}$, then we define the following $w_{11}, w_{12}, w_{21}, w_{22} \in \mathbb{C}$ in Fig. 1

$$w_{11} := r_2 \bar{\epsilon}_3 + w_{\mathbb{I}\mathbb{U}} = \left(r_2 \cos\left(\frac{3}{4}\pi\right) + a_0, r_2 \sin\left(\frac{3}{4}\pi\right) + b_0\right), \quad (14)$$



$$\begin{aligned}
w_{1U} &: (a_0, b_0) \\
w_{11} &: \left(-\frac{\sqrt{2}}{2}r_2 + a_0, \frac{\sqrt{2}}{2}r_2 + b_0\right) \\
w_{12} &: (-r_2 + a_0, b_0) \\
w_{1L} &: (a_0, -b_0) \\
w_{21} &: \left(\frac{\sqrt{2}}{2}r_2 + a_0, -\frac{\sqrt{2}}{2}r_2 - b_0\right) \\
w_{22} &: (r_2 + a_0, -b_0) \\
r_2 &: \text{radius of arcs centered at } w_{1U} \text{ and } w_{1L} \\
a_0 &= a_{0I} \text{ (between 0 and 1) and } r_2 > 0
\end{aligned}$$

Note: $b_0 > 0$

Figure 1: Location of complex numbers (indicated at the bottom of the figure) and contours in the complex w -plane. Contour C_1 passes through w_{11} , w_{1U} and w_{12} as in the integral directions \searrow , \leftarrow and \uparrow , in order. Contour C_2 passes through w_{1L} , w_{21} and w_{22} as in the directions \searrow , \uparrow and \leftarrow , in order. Symbols $I_{1\searrow}$, $I_{1\leftarrow}$ and $I_{1\phi}$ (given by Eq. (26)) are integrals of I_N (of Eq. (9)) along C_1 , while $I_{2\searrow}$, $I_{2\phi}$ and $I_{2\leftarrow}$ (given by Eq. (31)) are integrals of I_P along C_2 . ($I_{1\phi}$ and $I_{2\phi}$ are angular integrals around w_{1U} and w_{1L} , respectively.)

$$w_{12} := r_2(-1, 0) + w_{1U} = (-r_2 + a_0, b_0), \quad (15)$$

$$w_{21} := -r_2\bar{\varepsilon}_3 + w_{1L} = \left(r_2 \cos\left(-\frac{1}{4}\pi\right) + a_0, r_2 \sin\left(-\frac{1}{4}\pi\right) - b_0\right), \quad (16)$$

$$w_{22} := r_2(1, 0) + w_{1L} = (r_2 + a_0, -b_0). \quad (17)$$

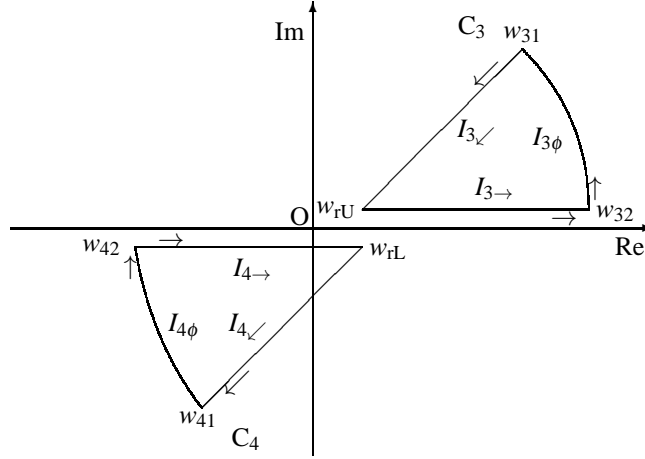
In a similar way, using a_0 ($a_0 = a_{0r}$ for I_P in Eq. (10)) and b_0 (with $b_0 > 0$) defined in the sentence above Eq. (11) and letting $w_{rU}, w_{rL} \in \mathbb{C}$, then we define the points in Fig. 2 as

$$w_{rU} := (a_0, b_0), \quad (18)$$

$$w_{rL} := (a_0, -b_0). \quad (19)$$

Additionally, let $\bar{\varepsilon} \in \mathbb{C}$ be defined by

$$\bar{\varepsilon} := \exp\left(\frac{1}{4}\pi i\right), \quad (20)$$



$$w_{rU}: (a_0, b_0)$$

$$w_{31}: (\frac{\sqrt{2}}{2}r_2 + a_0, \frac{\sqrt{2}}{2}r_2 + b_0)$$

$$w_{32}: (r_2 + a_0, b_0)$$

$$w_{rL}: (a_0, -b_0)$$

$$w_{41}: (-\frac{\sqrt{2}}{2}r_2 + a_0, -\frac{\sqrt{2}}{2}r_2 - b_0)$$

$$w_{42}: (-r_2 + a_0, -b_0)$$

$$r_2: \text{radius of arcs centered at } w_{rU} \text{ and } w_{rL}$$

$$a_0 = a_{0r} \text{ (between 0 and 1) and } r_2 > 0$$

Note: $b_0 > 0$

Figure 2: Location of complex numbers (indicated at the bottom of the figure) and contours in the complex w -plane. Contour C_3 passes through w_{31} , w_{rU} and w_{32} as in the integral directions \swarrow , \rightarrow and \uparrow , in order. Contour C_4 passes through w_{rL} , w_{41} and w_{42} as in the directions \swarrow , \uparrow and \rightarrow , in order. Symbols $I_{3\swarrow}$, $I_{3\rightarrow}$ and $I_{3\phi}$ (given by Eq. (33)) are integrals of I_P (of Eq. (10)) along C_3 , while $I_{4\swarrow}$, $I_{4\phi}$ and $I_{4\rightarrow}$ (given by Eq. (35)) are integrals of I_P along C_4 . ($I_{3\phi}$ and $I_{4\phi}$ are angular integrals around w_{rU} and w_{rL} , respectively.)

and let $r_2 \in \mathbb{R}$ with $r_2 > 0$ be the radius of arcs centered at w_{rU} and w_{rL} , then we further define the following $w_{31}, w_{32}, w_{41}, w_{42} \in \mathbb{C}$ in Fig. 2

$$w_{31} := r_2 \bar{e} + w_{rU} = (r_2 \cos(\frac{1}{4}\pi) + a_0, r_2 \sin(\frac{1}{4}\pi) + b_0), \quad (21)$$

$$w_{32} := r_2(1, 0) + w_{rU} = (r_2 + a_0, b_0), \quad (22)$$

$$w_{41} := -r_2 \bar{e} + w_{rL} = (r_2 \cos(-\frac{3}{4}\pi) + a_0, r_2 \sin(-\frac{3}{4}\pi) - b_0), \quad (23)$$

$$w_{42} := r_2(-1, 0) + w_{rL} = (-r_2 + a_0, -b_0). \quad (24)$$

Definition 3. Using the definitions of I_N , w_{1U} , w_{11} and w_{12} in Eqs. (9), (11), (14) and (15), respectively, and with the help of Fig. 1, we define the integrals and contours as

$$\oint_{C_1} dw I_N := I_{1\searrow} + I_{1\leftarrow} + I_{1\phi}, \quad (25)$$

where contour C_1 passes through w_{11} , w_{1U} and w_{12} as in the integral directions \searrow , \leftarrow and \uparrow , in order in Fig. 1, and

$$I_{1\searrow} := \int_{w_{11}}^{w_{1U}} dw I_N, \quad I_{1\leftarrow} := \int_{w_{1U}}^{w_{12}} dw I_N, \quad I_{1\uparrow} := \int_{w_{12}}^{w_{11}} dw I_N. \quad (26)$$

In the last integral above, let $\phi_1 \in \mathbb{R}$ be an angle of circular coordinates (centered at w_{1U}) of the integral variable, where the angle is measured counterclockwise from the axis parallel to the real axis, denoted as

$$w = r_2 \exp(i\phi_1) + w_{1U} = (r_2 \cos(\phi_1), r_2 \sin(\phi_1)) + (a_0, b_0), \quad (27)$$

with its derivative

$$\frac{dw}{d\phi_1} = ir_2 \exp(i\phi_1). \quad (28)$$

Then the last integral in Eq. (26) is defined by

$$I_{1\uparrow} = \int_{w_{12}}^{w_{11}} dw I_N := \int_{\pi}^{\frac{3}{4}\pi} d\phi_1 \frac{dw}{d\phi_1} I_N. \quad (29)$$

In a similar way, let $\phi_2, \phi_3, \phi_4 \in \mathbb{R}$ be angles of circular coordinates centered at w_{1L} , w_{rU} and w_{rL} , respectively. Using I_N , I_P (Eqs. (9), (10)), w_{1U} , w_{1L} (Eqs. (11), (12)), w_{21} , w_{22} (Eqs. (16), (17)), w_{rU} , w_{rL} (Eqs. (18), (19)) and w_{31} - w_{42} (Eqs. (21)-(24)), and with the help of Figs. 1-2, the other integrals and contours are defined by

$$\oint_{C_2} dw I_N := I_{2\searrow} + I_{2\phi} + I_{2\leftarrow}, \quad (30)$$

$$\left(\text{contour } C_2 \text{ passes through } w_{1L}, w_{21} \text{ and } w_{22} \text{ as in the directions } \searrow, \uparrow \text{ and } \leftarrow, \text{ in order in Fig. 1, and} \right. \\ \left. I_{2\searrow} := \int_{w_{1L}}^{w_{21}} dw I_N, \quad I_{2\phi} := \int_{w_{21}}^{w_{22}} dw I_N := \int_{-\frac{1}{4}\pi}^0 d\phi_2 \frac{dw}{d\phi_2} I_N \text{ with } w = r_2 \exp(i\phi_2) + w_{1L}, \quad I_{2\leftarrow} := \int_{w_{22}}^{w_{1L}} dw I_N \right), \quad (31)$$

$$\oint_{C_3} dw I_P := I_{3\swarrow} + I_{3\rightarrow} + I_{3\phi}, \quad (32)$$

$$\left(\text{contour } C_3 \text{ passes through } w_{31}, w_{rU} \text{ and } w_{32} \text{ as in the directions } \swarrow, \rightarrow \text{ and } \uparrow, \text{ in order in Fig. 2, and} \right. \\ \left. I_{3\swarrow} := \int_{w_{31}}^{w_{rU}} dw I_P, \quad I_{3\rightarrow} := \int_{w_{rU}}^{w_{32}} dw I_P, \quad I_{3\phi} := \int_{w_{32}}^{w_{31}} dw I_P := \int_0^{\frac{1}{4}\pi} d\phi_3 \frac{dw}{d\phi_3} I_P \text{ with } w = r_2 \exp(i\phi_3) + w_{rU} \right), \quad (33)$$

$$\oint_{C_4} dw I_P := I_{4\swarrow} + I_{4\phi} + I_{4\rightarrow}, \quad (34)$$

$$\left(\text{contour } C_4 \text{ passes through } w_{rL}, w_{41} \text{ and } w_{42} \text{ as in the directions } \swarrow, \uparrow \text{ and } \rightarrow, \text{ in order in Fig. 2, and} \right. \\ \left. I_{4\swarrow} := \int_{w_{rL}}^{w_{41}} dw I_P, \quad I_{4\phi} := \int_{w_{41}}^{w_{42}} dw I_P := \int_{-\frac{3}{4}\pi}^{-\pi} d\phi_4 \frac{dw}{d\phi_4} I_P \text{ with } w = r_2 \exp(i\phi_4) + w_{rL}, \quad I_{4\rightarrow} := \int_{w_{42}}^{w_{rL}} dw I_P \right). \quad (35)$$

We then present the lemma concerning the transformation mentioned at the beginning of this section.

Lemma 1. Let a_0 and b_0 be numbers that specify the x and y coordinates of w_{1L} , w_{1U} , w_{rU} and w_{rL} as expressed in Eqs. (11)-(12) and (18)-(19) with the help of Figs. 1-2. Let r_2 be the radius of the contour for the circular integrals indicated in Eqs. (27), (31), (33) and (35). Using the integrals $I_{1\searrow}$, $I_{2\searrow}$, $I_{3\swarrow}$, $I_{4\swarrow}$, $I_{1\leftarrow}$, $I_{2\leftarrow}$, $I_{3\rightarrow}$ and $I_{4\rightarrow}$ defined in Eqs. (26), (31), (33) and (35) with the help of Figs. 1-2, we derive

$$(I_{1\leftarrow} + I_{2\leftarrow} + I_{3\rightarrow} + I_{4\rightarrow}) = -(I_{1\searrow} + I_{2\searrow} + I_{3\swarrow} + I_{4\swarrow}) \\ \text{for a positive finite value of } a_0 \text{ in the limit } b_0 \rightarrow 0 \text{ which follows } r_2 \rightarrow \infty \text{ (with the relation such as } b_0 > 2/r_2). \quad (36)$$

The above equation implies in the $\varepsilon - \delta$ notation that

$$(\forall \varepsilon > 0)(\exists \delta > 0, \exists \delta_0 > 0)(\forall b_0, \forall r_2 \in \mathbb{R})$$

$$(\frac{2}{r_2} < \delta < b_0 < \delta_0 \Rightarrow |(I_{1\leftarrow} + I_{2\leftarrow} + I_{3\rightarrow} + I_{4\rightarrow}) + (I_{1\searrow} + I_{2\searrow} + I_{3\swarrow} + I_{4\swarrow})| < \varepsilon). \quad (37)$$

The sums on the left and right of the above equation take a definite finite value, stating that the sum of the integrals in the $-\bar{\varepsilon}_3\infty$ direction (indicated by \searrow) and in the $-\bar{\varepsilon}\infty$ direction (indicated by \swarrow) are transformed (converted) to the sum of the integrals along the line parallel to the real axis (indicated by \leftarrow and \rightarrow). The above relation holds for the exchange $z \leftrightarrow 1 - z$.

Proof. Using Eqs. (25), (26) and (30)-(35), each closed contour of the following integrals does not enclose the singularity at the coordinates origin in the complex plane when a_0 and b_0 (indicated in Eqs. (11)-(12) and (18)-(19) with the help of Figs. 1-2) are positive and finite. Then, the associated integrals along the closed contour vanish, due to the residue theorem, as expressed by

$$\oint_{C_1} dw I_N = \oint_{C_2} dw I_N = \oint_{C_3} dw I_P = \oint_{C_4} dw I_P = 0 \text{ for } b_0 > 0 \text{ (see also next 6 lines).} \quad (38)$$

(We note in advance that because of $b_0 > 0$ the above equation holds for a positive finite value of a_0 in the limit $b_0 \rightarrow 0$ that follows $r_2 \rightarrow \infty$ with the relation such as $b_0 > 2/r_2$, which implies that

$$(\forall \varepsilon > 0)(\exists \delta > 0, \exists \delta_0 > 0)(\forall b_0, \forall r_2 \in \mathbb{R})$$

$$(\frac{2}{r_2} < \delta < b_0 < \delta_0 \Rightarrow |\oint_{C_1} dw I_N| < \varepsilon, |\oint_{C_2} dw I_N| < \varepsilon, |\oint_{C_3} dw I_P| < \varepsilon, |\oint_{C_4} dw I_P| < \varepsilon). \quad (39)$$

Namely, because of $b_0 > 0$, no poles exist within the regions enclosed by the integral contours C_1, C_2, C_3, C_4 including contours, and the residue theorem works.)

Accordingly, with the use of Eqs. (25), (30), (32) and (34), we have

$$I_S = \oint_{C_1} dw I_N + \oint_{C_2} dw I_N + \oint_{C_3} dw I_P + \oint_{C_4} dw I_P$$

$$= (I_{1\searrow} + I_{1\leftarrow} + I_{1\phi}) + (I_{2\searrow} + I_{2\phi} + I_{2\leftarrow}) + (I_{3\swarrow} + I_{3\rightarrow} + I_{3\phi}) + (I_{4\swarrow} + I_{4\phi} + I_{4\rightarrow}) = 0. \quad (40)$$

Each of the above integral terms (such as $I_{1\searrow}, I_{1\leftarrow}$ and $I_{1\phi}$ in Eqs. (26), (31), (33) and (35)) has a definite finite value within a finite region that does not contain any singularities of the integrand when a_0 and b_0 (such as those in Eq. (11)) are positive and finite, and the order in the sum is changed to produce

$$(I_{1\searrow} + I_{2\searrow} + I_{3\swarrow} + I_{4\swarrow}) + (I_{1\leftarrow} + I_{2\leftarrow} + I_{3\rightarrow} + I_{4\rightarrow}) + (I_{1\phi} + I_{2\phi} + I_{3\phi} + I_{4\phi}) = 0. \quad (41)$$

For each of the last four terms $I_{1\phi}, I_{2\phi}, I_{3\phi}$ and $I_{4\phi}$ above, the real part of term $\mp \pi i w^2$ in the leading function $\exp(\mp \pi i w^2)$ (in the integrand I_N given by Eq. (9) for $I_{1\phi}$ and $I_{2\phi}$, such as that in (28)-(29), and the integrand I_P given by Eq. (10) for $I_{3\phi}$ and $I_{4\phi}$) yields $-2\pi|\text{Re}(w)||\text{Im}(w)|$ in the limit of $r_2 \rightarrow \infty$ (that is, for the large $|w|$) and the function $\exp(\mp \pi i w^2)$ vanishes for the large $|w|$. Then, we derive

$$I_{1\phi} = I_{2\phi} = I_{3\phi} = I_{4\phi} = 0 \quad \text{in the limit of } r_2 \rightarrow \infty \text{ (that is, for the large } |w| \text{) for } b_0 > 0. \quad (42)$$

$$\left(\begin{array}{l} \text{This implies that letting } b_0 > 0 \\ (\forall \varepsilon > 0)(\exists \delta > 0)(\forall r_2 \in \mathbb{R})(\frac{2}{r_2} < \delta \Rightarrow |I_{1\phi}| < \varepsilon, |I_{2\phi}| < \varepsilon, |I_{3\phi}| < \varepsilon, |I_{4\phi}| < \varepsilon). \end{array} \right) \quad (43)$$

In contrast, with the use of Eqs. (4) and (5), the sum of the first four terms $I_{1\searrow}, I_{2\searrow}, I_{3\swarrow}$ and $I_{4\swarrow}$ in Eq. (41) amounts to

$$(I_{1\searrow} + I_{2\searrow} + I_{3\swarrow} + I_{4\swarrow}) = \hat{\zeta}(1 - z)$$

for a positive finite value of a_0 in the limit $b_0 \rightarrow 0$ which follows $r_2 \rightarrow \infty$ (such as $b_0 > 2/r_2$). (44)

$$\left(\begin{array}{l} \text{This implies that} \\ (\forall \varepsilon > 0)(\exists \delta > 0, \exists \delta_0 > 0)(\forall b_0, \forall r_2 \in \mathbb{R})(\frac{2}{r_2} < \delta < b_0 < \delta_0 \Rightarrow |(I_{1\searrow} + I_{2\searrow} + I_{3\swarrow} + I_{4\swarrow}) - \hat{\zeta}(1-z)| < \varepsilon). \end{array} \right) \quad (45)$$

Thus, Eqs. (41)-(44) lead to

$$(I_{1\leftarrow} + I_{2\leftarrow} + I_{3\rightarrow} + I_{4\rightarrow}) = -(I_{1\searrow} + I_{2\searrow} + I_{3\swarrow} + I_{4\swarrow}) = -\hat{\zeta}(1-z) \quad \text{for a positive finite value of } a_0 \text{ in the limit } b_0 \rightarrow 0 \text{ which follows } r_2 \rightarrow \infty \text{ (such as } b_0 > 2/r_2). \quad (46)$$

$$\left(\begin{array}{l} \text{This means that} \\ (\forall \varepsilon > 0)(\exists \delta > 0, \exists \delta_0 > 0)(\forall b_0, \forall r_2 \in \mathbb{R}) \\ (\frac{2}{r_2} < \delta < b_0 < \delta_0 \Rightarrow |(I_{1\leftarrow} + I_{2\leftarrow} + I_{3\rightarrow} + I_{4\rightarrow}) + (I_{1\searrow} + I_{2\searrow} + I_{3\swarrow} + I_{4\swarrow})| \\ = |(I_{1\leftarrow} + I_{2\leftarrow} + I_{3\rightarrow} + I_{4\rightarrow}) + \hat{\zeta}(1-z)| < \varepsilon). \end{array} \right) \quad (47)$$

The above equation implies that the sum of the integrals in the $-\bar{\varepsilon}_3\infty$ direction (indicated by \searrow) and in the $-\bar{\varepsilon}_\infty$ direction (indicated by \swarrow) is transformed (converted) to the sum of the integrals along the line parallel to the real axis (indicated by \leftarrow and \rightarrow), owing to that the completed zeta function is regular. Namely, the integral of the completed zeta function along the line rotated from the real axis can be transformed to the integral along the line parallel to the real axis as a whole for convenience. We note that the above relationships hold for the exchange $z \leftrightarrow 1-z$, keeping the region $0 < \text{Re}(z) < 1$ under this exchange. \square

3 Proof of the Riemann's conjecture that the real part of all non-trivial zeros of the zeta function is 1/2

This section first converts the integral form of the completed zeta function expressed by Eq. (5) to the usual form to give Theorem 1. Then, in Theorem 2, we prove that the real part of all non-trivial zeros of the zeta function must be 1/2. As it is known that all non-trivial zeros of the zeta function exist in the region $0 < \text{Re}(z) < 1$ in literature [17, 18], we concentrate on this region. Furthermore, it is also known that the number of zeros (of the zeta function) with a real part of 1/2 is infinite [10]. The usual integral form of the zeta function is as follows.

Theorem 1. (the (third) integral form of the (completed) zeta function) Let $z, w \in \mathbb{C}$. Let $\hat{\zeta}(z)$ be the completed zeta function defined by Eq. (4). Let

$$\hat{\zeta}_1(z) := \pi^{-\frac{z}{2}} \Gamma(\frac{z}{2}) \int_{0\searrow 1} dw \frac{w^{-z} \exp(-\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)}, \quad (48)$$

and let

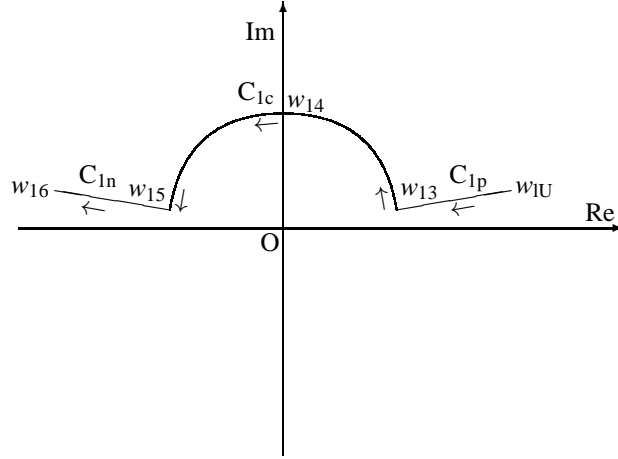
$$\hat{\zeta}_r(z) := \pi^{-\frac{1-z}{2}} \Gamma(\frac{1-z}{2}) \int_{0\swarrow 1} dw \frac{w^{z-1} \exp(\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)}, \quad (49)$$

in terms of the gamma function $\Gamma(z)$. Then $\hat{\zeta}(z)$ is expressed by

$$\hat{\zeta}(z) = \hat{\zeta}_1(z) + \hat{\zeta}_r(z), \quad (50)$$

which is called the (third) integral form of the (completed) zeta function.

Proof. From Eq. (5), the above completed zeta function $\hat{\zeta}(z)$ is obtained by the exchange $z \leftrightarrow 1-z$, where the region $0 < \text{Re}(z) < 1$ is kept under this exchange. The above region $0 < \text{Re}(z) < 1$ is consistent with the region $0 < \text{Re}(z) < 1$ we are considering in this paper. \square



$w_{1U}: (a_0, b_0)$
 $w_{13}: (a_0/2, b_0/2)$ $w_{14}: (0, r_1)$
 $w_{15}: (-a_0/2, b_0/2)$ $w_{16}: (-a_0, b_0)$
 C_{1p} : contour from w_{1U} to w_{13}
 C_{1c} : arc from w_{13} to w_{15} with radius r_1 centered at $(0, 0)$
 C_{1n} : contour from w_{15} to w_{16}
 $a_0 = a_{0l}$ (between 0 and 1) and $r_1 = [(a_0/2)^2 + (b_0/2)^2]^{1/2}$
 Note: $0 < b_0 \ll a_0$

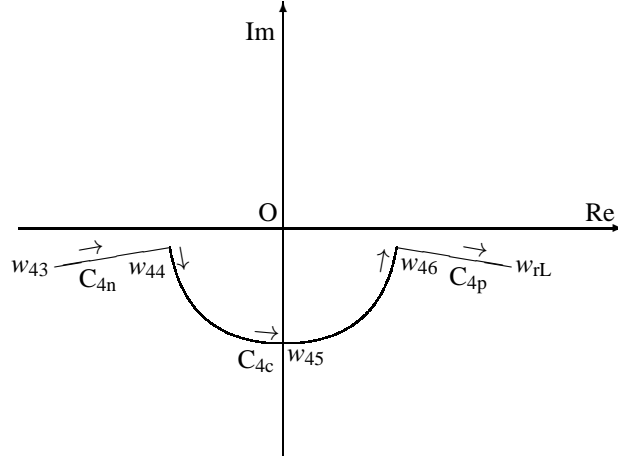
Figure 3: A part of the component parallel to the real axis in the contour C_1 in Fig. 1, deformed into the contour composed of C_{1p} , C_{1c} and C_{1n} (without disconnections from other parts) to bypass the origin of the complex w -plane. Locations of complex numbers in the complex plane are indicated at the bottom of the figure. The angle ϕ_c of circular coordinates centered at the origin $(0, 0)$ is measured counterclockwise from the real axis with the positive sign.

In addition, the completed zeta function satisfies the following known symmetry relation [1, 16, 15] for the exchange $z \leftrightarrow 1 - z$

$$\hat{\zeta}(z) = \hat{\zeta}(1 - z). \quad (51)$$

To derive the real part of non-trivial zeros of the zeta function, the present approach uses (in addition to the above symmetry given by Eq. (51)) the property (with merits) that a quantity in one term in a highly (attainable) symmetrized integral form generates a corresponding (paired) quantity in another term.

Here, for the Riemann-Siegel integral form, each part of the contour (for the integral variable $w \in \mathbb{C}$) parallel to the real axis in the contours C_1 and C_4 in Figs. 1-2 is further deformed to bypass the origin in the complex w -plane as in Figs. 3-4 according to Definition 4 below. Each contour part deformed as in Figs. 3-4 is connected (without disconnections) with the other remaining contours in Figs 1-2. The integrands have polynomials of w with the power of $z \in \mathbb{C}$, and the absolute value of the integrand is expressed using $|w|$ and the argument (angle) $\arg(w)$ in circular coordinates. Then, by making $\arg(w)$ independent of w near the origin of the w -plane, singularity analysis becomes simple. In Figs. 3-4, each linear contour employed has a constant $\arg(w)$, while $\arg(w)$ along each arc (of circle) contour is integrated to the circular integral. Therefore, singularity analysis focuses on the power for the polynomials of $|w|$. (We note that no pole exists around the pole located at the origin of the complex w -plane, when the concerned region is small.)



$$\begin{aligned}
w_{rL} &: (a_0, -b_0) \\
w_{43} &: (-a_0, -b_0) & w_{44} &: (-a_0/2, -b_0/2) \\
w_{45} &: (0, -r_1) & w_{46} &: (a_0/2, -b_0/2) \\
C_{4n} &: \text{contour from } w_{43} \text{ to } w_{44} \\
C_{4c} &: \text{arc from } w_{44} \text{ to } w_{46} \text{ with radius } r_1 \text{ centered at } (0,0) \\
C_{4p} &: \text{contour from } w_{46} \text{ to } w_{rL} \\
a_0 &= a_{0r} \text{ (between 0 and 1) and } r_1 = [(a_0/2)^2 + (b_0/2)^2]^{1/2} \\
\text{Note: } &0 < b_0 < a_0
\end{aligned}$$

Figure 4: A part of the component parallel to the real axis in the contour C_4 in Fig. 2, deformed into the contour composed of C_{4n} , C_{4c} and C_{4p} (without disconnections from other parts) to bypass the origin of the complex w -plane. Locations of complex numbers in the complex plane are indicated at the bottom of the figure. The angle ϕ_c of circular coordinates centered at the origin $(0,0)$ is measured counterclockwise from the real axis with the positive sign.

Definition 4. Let $b_0, a_0 \in \mathbb{R}$ with b_0 and a_0 being small ($b_0 > 0$ as indicated such as in Eq. (45); $b_0 < a_0$ as described in Note below Eq. (65)). Let $w_{1U} \in \mathbb{C}$ as defined in Definition 2 with Figs. 1-2. Letting $\phi_c \in \mathbb{R}$, we define the angle ϕ_c of circular coordinates centered at the origin $(0,0)$ of the complex w -plane, that is measured counterclockwise from the real axis with the positive sign. Let $r_1 \in \mathbb{R}$ (radius of the arc (of circle) contours defined below and in Figs. 3-4) such that $r_1 = a_0/2$. Let $w_{13}, w_{14}, w_{15}, w_{16} \in \mathbb{C}$, then, as indicated in Fig. 3, we define these numbers in the complex w -plane by

$$\begin{aligned}
w_{13} &:= \left(\frac{a_0}{2}, \frac{b_0}{2}\right), & w_{14} &:= (0, r_1), \\
w_{15} &:= \left(-\frac{a_0}{2}, \frac{b_0}{2}\right), & w_{16} &:= (-a_0, b_0),
\end{aligned} \tag{52}$$

where

$$r_1 := \left[\left(\frac{a_0}{2}\right)^2 + \left(\frac{b_0}{2}\right)^2 \right]^{1/2}. \tag{53}$$

Using the quantities given in Fig. 1, a part of the component, parallel to the real axis, of the contour C_1 in Fig. 1 is deformed into the contour composed of C_{1p} , C_{1c} and C_{1n} to bypass the coordinate origin as in Fig. 3. The contours in Fig. 3 are defined as follows.

- C_{1p} : line contour from w_{1U} to w_{13} in the direction of the arrow,
- C_{1c} : arc (of circle) contour from w_{13} to w_{15} via w_{14} (in each direction of the arrow) centered at the coordinate origin $(0,0)$ with the radius r_1 ,
- C_{1n} : line contour from w_{15} to w_{16} in the direction of the arrow.

Similarly, let $w_{1L} \in \mathbb{C}$ defined in Definition 2 with Figs. 1-2. Let $w_{43}, w_{44}, w_{45}, w_{46} \in \mathbb{C}$, then, as indicated in Fig. 4, we define these numbers by

$$\begin{aligned} w_{43} &:= (-a_0, -b_0), & w_{44} &:= \left(-\frac{a_0}{2}, -\frac{b_0}{2}\right), \\ w_{45} &:= (0, -r_1), & w_{46} &:= \left(\frac{a_0}{2}, -\frac{b_0}{2}\right), \end{aligned} \quad (54)$$

in the complex w -plane. Using quantities given in Fig. 2, a part of the component, parallel to the real axis, of the contour C_4 in Fig. 2 is deformed into the contour composed of C_{4n} , C_{4c} and C_{4p} to bypass the coordinate origin as in Fig. 4. The contours in Fig. 4 are defined as follows.

- C_{4n} : line contour from w_{43} to w_{44} in the direction of the arrow,
- C_{4c} : arc (of circle) contour from w_{44} to w_{46} via w_{45} (in each direction of the arrow) centered at the coordinate origin $(0,0)$ with the radius r_1 ,
- C_{4p} : line contour from w_{46} to w_{1L} in the direction of an arrow.

Definition 5. Let $w, z \in \mathbb{C}$, then with the definition

$$A_l := w^{-z}, \quad (55)$$

$$A_r := w^{z-1}, \quad (56)$$

integrand I_l along the above contours C_{1p} , C_{1c} and C_{1n} in Fig. 3 is defined as follows

$$I_l := A_l \frac{\exp(-\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)}, \quad (57)$$

whereas integrand I_r along the contours C_{4n} , C_{4c} and C_{4p} in Fig. 4 is defined as follows

$$I_r := A_r \frac{\exp(\pi i w^2)}{\exp(\pi i w) - \exp(-\pi i w)}. \quad (58)$$

Furthermore, integral I_{1c}^S along the above circular contour C_{1c} (in Fig. 3) is defined by

$$I_{1c}^S := \oint_{C_{1c}} dw I_l, \quad (59)$$

whereas I_{4c}^S along the circular contour C_{4c} (in Fig. 4) is defined by

$$I_{4c}^S := \oint_{C_{4c}} dw I_r. \quad (60)$$

(Note: The above I_l and I_r in Eqs. (57)-(58) correspond to I_N and I_P in Eqs. (9)-(10), respectively, if the multiplication factors are dropped from the integrands. Owing to Theorem 1 and Eq. (51), the complex numbers z and $1-z$ in these factors were exchanged.)

Lemma 2. Let $z, w \in \mathbb{C}$, $z_R = \text{Re}(z)$ and $\hat{\zeta}(z)$ be the completed zeta function in Eq. (50). Using $b_0, a_0, r_1, \phi_c \in \mathbb{R}$ with b_0, a_0 being small ($0 < b_0 \ll a_0$) in Definition 4 with Figs. 3-4, the integrand I_l (along the contours C_{1p} and C_{1n}) of $\hat{\zeta}_l(z)$ in Eq. (48) composing $\hat{\zeta}(z)$ has the form $|w|^{-z_R-1}$. Similarly, the integrand I_r (along the contours C_{4n} and C_{4p}) of $\hat{\zeta}_r(z)$ in Eq. (49) composing $\hat{\zeta}(z)$ has the form $|w|^{z_R-1-1}$. These forms have a singularity caused by the powers $-z_R-1$ and z_R-1-1 of $|w|$, respectively, with $0 < z_R < 1$ in the region near $w = 0$ (for the limit of $b_0 \rightarrow 0$ with $0 < b_0 \ll a_0$ followed by $a_0 \rightarrow 0$).

Proof. For small $|w|$ near the origin of the complex w -plane, it follows that

$$\exp(\mp iw^2) \approx 1, \quad (61)$$

$$\frac{1}{\exp(\pi iw) - \exp(-\pi iw)} \approx \frac{1}{2\pi iw} \approx \frac{1}{w}. \quad (62)$$

Then, (using $z_R = \text{Re}(z)$, $z_I = \text{Im}(z)$) the integrand I_l in Eq. (57) for small $|w|$ ($w \neq 0$ from Fig. 3) is written as follows

$$\begin{aligned} I_l &\approx \frac{A_l}{w} = w^{-z-1} = w^{-z_R-1} w^{-iz_I} = w^{-z_R-1} \exp\{\ln[w^{-iz_I}]\} \\ &= w^{-z_R-1} \exp\{-iz_I[\ln(|w|) + i \arg(w)]\} \\ &= w^{-z_R-1} \exp[-iz_I \ln(|w|)] \exp[z_I \arg(w)], \end{aligned} \quad (63)$$

where $\arg(w)$, which denotes argument of w , is restricted to the principal value given by

$$\arg(w) = \begin{cases} \varepsilon_p & \text{for the contour } C_{1p} \\ \pi - \varepsilon_p & \text{for the contour } C_{1n}, \end{cases} \quad (64)$$

with

$$\begin{aligned} \varepsilon_p &= \sin^{-1} \left[\frac{b_0}{(a_0^2 + b_0^2)^{1/2}} \right] \\ &= \arcsin \left[\frac{b_0}{(a_0^2 + b_0^2)^{1/2}} \right] \quad \text{for small } b_0, a_0 \text{ with } 0 < b_0 \ll a_0. \end{aligned} \quad (65)$$

(Note: We set such as $a_0 = b_0^{1/2}$, namely, $a_0^2 = b_0$ in the limit of $b_0 \rightarrow 0$ with $b_0 > 0$. Then, $b_0/(a_0^2 + b_0^2)^{1/2} = b_0^{1/2}/(1 + b_0)^{1/2} \rightarrow 0$ in the limit of $b_0 \rightarrow 0$. Concerning $a_0 = a_{0l}$ (in Figs. 1 and 3) and $a_0 = a_{0r}$ (in Figs. 2 and 4 with Definition 2), we can set such as $|a_{0l} - a_{0r}| < b_0^2$ (when a_{0l} and a_{0r} differ from each other), which is negligible compared to b_0 in the limit of $b_0 \rightarrow 0$, implying that $a_{0l} \approx a_{0r}$ in this limit.)

Therefore, using Eqs. (63)-(65), we derive in the limit of $b_0 \rightarrow 0$ ($0 < b_0 \ll a_0$)

$$\arg(w) = \begin{cases} 0 & \text{for the contour } C_{1p} \\ \pi & \text{for the contour } C_{1n}, \end{cases} \quad (66)$$

and

$$|I_l| = \begin{cases} |w|^{-z_R-1} & \text{for the contour } C_{1p} \\ |w|^{-z_R-1} \exp(z_I \pi) & \text{for the contour } C_{1n}. \end{cases} \quad (67)$$

Hence, disregarding the constant $\exp(z_I \pi)$ (see also Note below Eq. (87)), the integrand I_l has the form $|w|^{-z_R-1}$ with a singularity caused by the power $-z_R-1$ of $|w|$ with $0 < z_R < 1$. (Constants including z_I will be used for the z_I determination, which is beyond the scope of this paper.)

Meanwhile, by the replacement $z \rightarrow 1 - z$ in Eq. (63), that is,

$$z_R \rightarrow 1 - z_R, \quad z_I \rightarrow -z_I, \quad (68)$$

the integrand I_r in Eq. (58) for small $|w|$ ($w \neq 0$ from 4) is written as follows

$$I_r \approx w^{z_R-1-1} \exp[i z_I \ln(|w|)] \exp[-z_I \arg(w)], \quad (69)$$

where, using ε_p in Eq. (65) with Fig 4 (in comparison with Fig. 3), $\arg(w)$ in this case is given by

$$\arg(w) = \begin{cases} -\varepsilon_p & \text{for the contour } C_{4p} \\ -\pi + \varepsilon_p & \text{for the contour } C_{4n}. \end{cases} \quad (70)$$

Therefore, with the use of Eq. (69), we have, in the limit of $b_0 \rightarrow 0$ ($b_0 < a_0$),

$$\arg(w) = \begin{cases} 0 & \text{for the contour } C_{4p} \\ -\pi & \text{for the contour } C_{4n}, \end{cases} \quad (71)$$

and

$$|I_r| = \begin{cases} |w|^{z_R-1-1} & \text{for the contour } C_{4p} \\ |w|^{z_R-1-1} \exp(z_I \pi) & \text{for the contour } C_{4n}. \end{cases} \quad (72)$$

Hence, disregarding the constant $\exp(z_I \pi)$ (refer also Note below Eq. (87)), the integrand I_r has the form $|w|^{z_R-1-1}$ with a singularity caused by the power $z_R - 1 - 1$ of $|w|$ for $0 < z_R < 1$. \square

Lemma 3. Let $z, w \in \mathbb{C}$, $z_R = \text{Re}(z)$ and $\hat{\zeta}(z)$ be the completed zeta function in Eq. (50). With the use of $b_0, a_0, r_1, \phi_c \in \mathbb{R}$ with b_0, a_0 being small ($0 < b_0 < a_0$) in Definition 4 with Figs. 3-4, the circular integral I_{1c}^S (in Eq. (59) along the contour C_{1c}) of $\hat{\zeta}_l(z)$ in Eq. (48) composing $\hat{\zeta}(z)$ has the form $|w|^{-z_R}$. Similarly, the circular integral I_{4c}^S (in Eq. (60) along the contour C_{4c}) of $\hat{\zeta}_r(z)$ in Eq. (49) composing $\hat{\zeta}(z)$ has the form $|w|^{z_R-1}$. These forms have singularities caused by the powers $-z_R$ and $z_R - 1$ of $|w|$, respectively, with $0 < z_R < 1$ in the region near $w = 0$ (for the limit of $b_0 \rightarrow 0$ with $0 < b_0 < a_0$ followed by $a_0 \rightarrow 0$).

Proof. With the use of Eqs. (61)-(62) and Eq. (65), the circular integral I_{1c}^S in Eq. (59) (with Eq. (55) along the contour C_{1c} in Fig. 3) for small $|w|$ near the origin in the complex w -plane is expressed as

$$\begin{aligned} I_{1c}^S &\approx \oint_{C_{1c}} dw \frac{A_1}{w} = \oint_{C_{1c}} dw w^{-z-1} \\ &= \int_{\varepsilon_p}^{\pi-\varepsilon_p} d\phi_c \frac{dw}{d\phi_c} w^{-z-1}. \end{aligned} \quad (73)$$

Using $|w| = r_1$ on the contour C_{1c} (and on C_{4c}) and

$$w = |w| \exp(i\phi_c) = r_1 \exp(i\phi_c), \quad (74)$$

with

$$\frac{dw}{d\phi_c} = i r_1 \exp(i\phi_c) = i w, \quad (75)$$

we obtain

$$I_{1c}^S = \int_{\varepsilon_p}^{\pi-\varepsilon_p} d\phi_c(i) w^{-z}. \quad (76)$$

The integrand of Eq. (76) with $z_R = \text{Re}(z)$ and $z_I = \text{Im}(z)$ produces

$$\begin{aligned}
iw^{-z} &= iw^{-z_R - iz_I} = iw^{-z_R} \exp[\ln(w^{-iz_I})] \\
&= iw^{-z_R} \exp[-iz_I \ln(w)] = iw^{-z_R} \exp\{(-iz_I)[\ln(|w|) + i \arg(w)]\} \\
&= iw^{-z_R} \exp[-iz_I \ln(|w|)] \exp[z_I \arg(w)].
\end{aligned} \tag{77}$$

Then, using Eqs. (65), (74) (with $|w| = r_1$ on the contour C_{1c} and on C_{4c}) and (77), the integral in Eq. (76) becomes

$$\begin{aligned}
I_{1c}^S &= \int_{\varepsilon_p}^{\pi - \varepsilon_p} d\phi_c(i) w^{-z_R} \exp[-iz_I \ln(|w|)] \exp(z_I \phi_c) \\
&= \int_{\varepsilon_p}^{\pi - \varepsilon_p} d\phi_c(i) |w|^{-z_R} \exp[-iz_R \phi_c] \exp[-iz_I \ln(|w|)] \exp(z_I \phi_c) \\
&= i |w|^{-z_R} \exp[-iz_I \ln(|w|)] \frac{\exp[(-iz_R + z_I)(\pi - \varepsilon_p)] - \exp[(-iz_R + z_I)\varepsilon_p]}{-iz_R + z_I}.
\end{aligned} \tag{78}$$

From Eq. (65), we have, in the limit of $b_0 \rightarrow 0$ ($0 < b_0 < a_0$),

$$I_{1c}^S \approx i |w|^{-z_R} \exp[-iz_I \ln(|w|)] \frac{\exp[(-iz_R + z_I)\pi] - 1}{-iz_R + z_I}. \tag{79}$$

Therefore,

$$|I_{1c}^S| = |w|^{-z_R} \frac{|\exp[(-iz_R + z_I)\pi] - 1|}{|-iz_R + z_I|}. \tag{80}$$

Hence, disregarding the constant $|\exp[(-iz_R + z_I)\pi] - 1|/|-iz_R + z_I|$ (see also Note below Eq. (87)), the integral I_{1c}^S has the form $|w|^{-z_R} = r_1^{-z_R}$ with the singularity caused by the power $-z_R$ of $|w|$ for $0 < z_R < 1$.

Whereas, the circular integral I_{4c}^S in Eq. (60) (along the contour C_{4c} in Fig. 4) for small $|w|$ near the origin in the complex w -plane is calculated by the replacement $z \rightarrow 1 - z$ with Figs. 3-4 in Eq. (78), that is,

$$z_R \rightarrow 1 - z_R, \quad -z_R \rightarrow z_R - 1, \quad z_I \rightarrow -z_I, \quad \int_{\varepsilon_p}^{\pi - \varepsilon_p} d\phi_c \rightarrow \int_{-\pi - \varepsilon_p}^{-\varepsilon_p} d\phi_c, \tag{81}$$

yielding

$$\begin{aligned}
I_{4c}^S &= \int_{-\pi - \varepsilon_p}^{-\varepsilon_p} d\phi_c(i) w^{z_R - 1} \exp[iz_I \ln(|w|)] \exp(-z_I \phi_c) \\
&= \int_{-\pi - \varepsilon_p}^{-\varepsilon_p} d\phi_c(i) |w|^{z_R - 1} \exp[i(z_R - 1)\phi_c] \exp[iz_I \ln(|w|)] \exp(-z_I \phi_c) \\
&= i |w|^{z_R - 1} \exp[iz_I \ln(|w|)] \frac{\exp\{[i(z_R - 1) - z_I](-\varepsilon_p)\} - \exp\{[i(z_R - 1) - z_I](-\pi + \varepsilon_p)\}}{i(z_R - 1) - z_I}.
\end{aligned} \tag{82}$$

Using Eq. (65), the above equation becomes, in the limit of $b_0 \rightarrow 0$ ($0 < b_0 < a_0$)

$$I_{4c}^S \approx i |w|^{z_R - 1} \exp[i(z_I) \ln(|w|)] \frac{1 - \exp\{[i(z_R - 1) - z_I](-\pi)\}}{i(z_R - 1) - z_I}. \tag{83}$$

Therefore,

$$|I_{4c}^S| = |w|^{z_R - 1} \frac{|\exp\{[i(z_R - 1) - z_I](-\pi)\} - 1|}{|i(z_R - 1) - z_I|}. \tag{84}$$

Hence, disregarding the constant $|\exp\{[i(z_R - 1) - z_I](-\pi)\} - 1|/|i(z_R - 1) - z_I|$ (see also Note below Eq. (87)), the integral I_{4c}^S has the form $|w|^{z_R - 1} = r_1^{z_R - 1}$ with the singularity caused by the power $z_R - 1$ of $|w|$ for $0 < z_R < 1$. \square

We then prove the following theorem, which completes the proof of the Riemann hypothesis.

Theorem 2. *Let $z \in \mathbb{C}$ and let $z_R = \text{Re}(z)$. Let $\hat{\zeta}(z)$ be the completed zeta function given in Theorem 1. To satisfy $\hat{\zeta}(z) = 0$, the real component (real part) z_R of the non-trivial zeros of the (completed) zeta function must take the following value*

$$z_R = \frac{1}{2}, \quad (85)$$

which is a proof of the Riemann hypothesis.

Proof. Both the gamma function $\Gamma(z)$ and zeta function $\zeta(z)$ in the region $0 < \text{Re}(z) < 1$ under consideration are regular without a singularity as described below Eq. (5). The completed zeta function $\hat{\zeta}(z)$ defined by Eq. (4), which is a product between $\Gamma(z)$ and $\zeta(z)$, is also regular in the region $0 < \text{Re}(z) < 1$, as described below Eq. (5). As mentioned above in Section 1 (Introduction), the function $\hat{\zeta}(z)$ does not depend on a specific value of the parameter a_0 (between 0 and 1), which specifies the intersection point of the integral line and the real axis, owing to the residue theorem. However, the integrands of the elements $\hat{\zeta}_l(z)$ in Eq. (48) and $\hat{\zeta}_r(z)$ in Eq. (49) composing $\hat{\zeta}(z)$ in Eq. (50) contain the following singularity near $w = 0$, only in the case of $a_0 \rightarrow 0$.

We note that the completed zeta function $\hat{\zeta}(z)$ does not depend on a specific value of a_0 between 0 and 1 due to the residue theorem, as was described below Eq. (5), whereas the singularity of each element $\hat{\zeta}_l(z)$ and $\hat{\zeta}_r(z)$ of $\hat{\zeta}(z)$ depends on a_0 . However, these singularities and the dependence of the elements $\hat{\zeta}_l(z)$ and $\hat{\zeta}_r(z)$ on a_0 adequately (incompletely) cancel each other by remaining a finite value for $\hat{\zeta}(z) \neq 0$, because the integral directions projected to the line parallel to the real axis for $\hat{\zeta}_l(z)$ and for $\hat{\zeta}_r(z)$ are opposite, and result in the finite completed zeta function $\hat{\zeta}(z)$ without the dependence on a_0 .

In contrast, for $\hat{\zeta}(z) = 0$, the singularities must exactly cancel each other. Let z_R and z_I be the real and imaginary components of z , respectively. Lemma 2 states that (near $w = 0$ in the case of $b_0 \rightarrow 0$ with $0 < b_0 < a_0$ followed by $a_0 \rightarrow 0$), from Eq. (67) (refer also Note below Eq. (87)), the integrand I_l (in Eq. (57)) of $\zeta_l(z)$ in Eq. (48) has the following power, which leads to the singularity,

$$|I_l| \approx |w|^{-z_R-1} \quad (\text{integrand on contours } C_{1p} \text{ and } C_{1n} \text{ in Fig. 3}), \quad (86)$$

while, from Eq. (72), the integrand I_r (in Eq. (58)) of $\zeta_r(z)$ in Eq. (49) has the following power

$$|I_r| \approx |w|^{z_R-1-1} \quad (\text{integrand on contours } C_{4p} \text{ and } C_{4n} \text{ in Fig. 4}). \quad (87)$$

(We disregarded the constant factors in Eq. (67) for the contours C_{1p} and C_{1n} and those in Eq. (72) for the contours C_{4n} and C_{4p} .)

(Note: Letting $\alpha_1, \alpha_2, w \in \mathbb{C}$ and $\beta_1, \beta_2 \in \mathbb{R}$ with $0 < \beta_1 < \beta_2$, if $|w| < (|\alpha_2|/|\alpha_1|)|w|^{-\beta_2})^{\beta_1}$ for small $|w|$, then we obtain $|\alpha_1||w|^{-\beta_1} < |\alpha_2||w|^{-\beta_2}$, which implies that these two terms with different order powers cannot cancel each other for sufficiently small $|w|$, as used below.)

Meanwhile, Lemma 3 states that (near $w = 0$ in the case of $b_0 \rightarrow 0$ with $0 < b_0 < a_0$ followed by $a_0 \rightarrow 0$), from Eq. (80), the integral I_{lc}^S (in Eq. (59)) for $\zeta_l(z)$ in Eq. (48) has the following power

$$|I_{lc}^S| \approx |w|^{-z_R} \quad (\text{integral along contour } C_{1c} \text{ in Fig. 3}), \quad (88)$$

while, from Eq. (84), the integral I_{4c}^S (in Eq. (60)) for $\zeta_r(z)$ in Eq. (49) has the following power

$$|I_{4c}^S| \approx |w|^{z_R-1} \quad (\text{integral along contour } C_{4c} \text{ in Fig. 4}). \quad (89)$$

(We also disregarded the constant factor in Eq. (80) for the contour C_{1c} and that in Eq. (84) for the contour C_{4c} .)

To satisfy $\hat{\zeta}(z) = 0$, these singularities should have an identical order power of $|w|$ and exactly cancel each other in the oppositely directed integrals projected to the line parallel to the real axis with additional deformations (as described in detail in Lemmas 1-3). We then derive the main concluding relation, from Eqs. (86)-(87), that

$$-z_R - 1 = z_R - 1 - 1, \quad (90)$$

which is reduce to

$$-z_R = z_R - 1, \quad (91)$$

that also equivalently works on Eqs. (88)-(89) as a whole, and this relation finally results in the expected requirement

$$z_R = \frac{1}{2}, \quad (92)$$

stating that all non-trivial zeros of the (completed) zeta function have real component (part) of $1/2$, which is the proof of the Riemann hypothesis.

Namely, considering that a_0 specifies the contour, $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall a_0 \in \mathbb{R} \text{ with } 0 < a_0 < 1)(a_0 < \delta \Rightarrow |z_R - \frac{1}{2}| < \varepsilon)$. Furthermore, the completed zeta function $\hat{\zeta}(z)$ is a product between the gamma function $\Gamma(z)$ and zeta function $\zeta(z)$ as in Eq. (4), and the functions $\hat{\zeta}(z)$, $\Gamma(z)$ and $\zeta(z)$ are regular in the region $0 < \text{Re}(z) < 1$. Then, the solution of $\hat{\zeta}(z) = 0$ (which is independent of the contour specified by a_0 unlike $\hat{\zeta}_l(z)$ and $\hat{\zeta}_r(z)$ composing $\hat{\zeta}(z)$) satisfies $\zeta(z) = 0$ and vice versa. Thus, we have completed the proof of the Riemann hypothesis. \square

Remark 1. We here show the implication of the above process and derived solution. In the integrands of the elements $\hat{\zeta}_l(z)$ and $\hat{\zeta}_r(z)$ composing the completed zeta function $\hat{\zeta}(z)$, the singularities appear in the oppositely directed integrals (projected to the line parallel to the real axis) of polynomials. Furthermore, the completed zeta function is symmetrized with respect to $\text{Re}(z) = 1/2$. The functions $\hat{\zeta}_l(z)$ and $\hat{\zeta}_r(z)$ adequately (by incompletely remaining a finite value) cancel each other for $\text{Re}(z) \neq 1/2$, while this cancellation is complete only for $\text{Re}(z) = 1/2$, leading to $\hat{\zeta}(z) = 0$.

In conclusion, we have inspected in detail the singularities of the integral form of the completed zeta function. For $\hat{\zeta}(z) = 0$ (that is, $\zeta(z) = 0$), the singularities of the integral along the two rotated integral contours (lines) are required to exactly cancel each other, when the intersection points between the integral lines and the real axis approach the coordinate origin. This approach of the intersection points to the origin is possible because of the arbitrariness of the intersection points owing to the residue theorem. Thus, we have shown that the real part of all non-trivial zeros of the zeta function is $1/2$, which is the proof of the Riemann hypothesis.

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